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On Free Products of Isomorphic Free Groups with a Single Finitely Generated Amalgamated Subgroup

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The residual finiteness of the free product of isomorphic free groups with a single finitely generated amalgamated subgroup will be shown. The method used is that of reference [*1*], extended by use of theorem of Marshall Hall. The theorem generalizes a result of G. Baumslag [*3*], on the free product of free groups with a cyclic amalgamated subgroup.

A few lemmas will be needed in the proof of the theorem. The proof of the first lemma appears here, while the others appear in reference [*1*].

LEMMA 1. *Let F be a free group, g an element of F and H a finitely generated subgroup of F not containing g. Under these conditions, there is a normal subgroup N of F of finite index in F such that the coset gH does not have an element in common with N.*

Proof. Let H^1 be a subgroup of finite index in F containing all the generators of H but not containing g. According to M. Hall [*2*], such a subgroup H^1 exists. Let N be the intersection of all the distinct conjugates of H^1 in F. Every conjugate of H^1 is of the same index as H^1 in F, and there are only finitely many distinct conjugates of H^1 in F. Since N is the intersection of finitely many subgroups of finite index, N is of finite index in F. Suppose gH has an element in common with N. Since N is contained in H^1, H^1 would have an element in common with gH, and since H^1 contains H, H^1 would contain g, contrary to hypothesis. Thus N is a subgroup of F with the properties stated in the lemma.

LEMMA 2. *Let A_i, $i = 1, 2, \ldots$ be isomorphic groups and let f_i be isomorphisms such that $f_i A_i = A_i$. Let H_1 be a subgroup of A_1 and let $H_i = f_i H_1$ be subgroups of A_i. Let G be the free product of the A_i with the subgroups H_i amalgamated under isomorphisms $H_i = f_i f_j^{-1} H_j$. Let N_1 be a normal subgroup of A_1 and let $N_i = f_i N_i$. Let G^t be the free product of the groups A_i/N_i with the subgroups $H_i/H_i \cap N_i$ amalgamated under isomorphisms induced by $f_i f_j^{-1}$.*

Then

2.1 *A_1 is the image of G under a homomorphism of which acts as f_i^{-1} on elements of A_i.*

2.2 *G^t is the image of G under a homomorphism of which acts as the natural map $A_i \to A_i/N_i$ on elements of A_i.*

LEMMA 3. *The free product of isomorphic finite groups with a single amalgamated subgroup is Π_c.*

The Π_c property is defined as follows: A group G is Π_c if, and only if, for every ordered pair g_1, g_2 of elements of G the following alternative holds: Either $g_1 = g_2{}^t$ for some integer t or there is a normal subgroup N of finite index in G such that $g_1 \not\equiv g_2{}^t \bmod N$ for all integers t. A group G is residually finite if and only if for every nonidentity element g of G there is a normal subgroup N of finite index in G not containing g. Clearly, if a group is Π_c it is residually finite.[1]

LEMMA 4. *Let g_2 be a cyclically reduced element of G. An element g_1 of G is in the cyclic subgroup generated by g_2 only under the following conditions:*

a) *If g_2 has syllable length one, say $g_2 = a_1$ then $g_2 = a_1{}^t$ for some integer t.*

b) *If g_2 has syllable length greater than one, say $g_2 = a_1 \cdots a_m$ then $g_1 = b_1 \cdots b_n$ in reduced form with $n = km$ for some integer $k > 0$. In this case $g = g_2{}^t$ only if $t = \pm k$.* If $g = g_2{}^k$ *then*

$$
\begin{aligned}
&a_1^{-1} b_1 = c_1 \\
&a_2^{-1} c_1 b_2 = c_2 \\
&\quad\vdots \\
&a_i^{-i} c_{i-1} b_i = c_i \qquad a_{j+m} = a_j \\
&\quad\vdots \\
&a_{km}^{-1} c_{km-1} b_{km} = c_{km} = 1
\end{aligned}
$$

where c_i are in the amalgamated subgroup. If $g_1 = g_2^{-k}$, similar equalities hold with a_m^{-1} replacing a_1, a_{m-1}^{-1} replacing a_2, etc.

THEOREM. *The free product of isomorphic free groups with a single finitely generated subgroup amalgamated is Π_c.*

[1] A theorem in reference [*1*] together with an example by G. Baumslag show that Π_c is stronger than residual finiteness.

Let G be the free product of isomorphic free groups $A_1, \ldots,$ where $A_i = \varphi_i(A_1)$, φ_i is on isomorphism and $\varphi_1 = 1$. Let B_1 be a finitely generated subgroup of A_1 and let $B_i = \varphi_i(B_1)$. G is to be the free product of the A_i with the subgroups B_i amalgamated under the isomorphisms $B_i = \varphi_i\varphi_j^{-1}(B_j)$.

Case I. Let g_1 and g_2 be two elements of G such that $g_1 \neq g_2{}^t$ for all t and g_2 is cyclically reduced. Suppose $g_1 \neq g_2{}^t$ for all t is implied by the syllable length of g_1; that is, if g_2 has syllable length 1, g_1 has syllable length greater than one; if g_2 has syllable length m greater than one, then the syllable length of g_1 is not divisible by m.

We will find a Π_c image group, $X(G)$ the free product of finite images of the A_i with a single subgroup amalgamated, in which the images $X(g_1)$ and $X(g_2)$ have the same syllable lengths as g_1 and g_2 respectively, and so $X(g_1) \neq X(g_2)^t$ for all t. Let $g_2 = a_1 \cdots a_m$, $g_1 = b_1 \cdots b_n$. If neither g_1 *nor* g_2 is in the amalgamated subgroup, no syllable is in the amalgamated subgroup. Let $a_i \in A_{n_i}$, $b_i \in A_{m_i}$ and let $\bar{N}_i$ be a normal subgroup of A_1 such that $\varphi_{n_i}^{-1}(a_i)\, B_1 \cap \bar{N}_i$ is empty and $A_1/\bar{N}_i$ is finite. Let $\bar{M}_i$ be a normal subgroup of A_1 such that $\varphi_{m_i}^{-1}(b_i)\, B_1 \cap \bar{M}_i$ is empty. Let $N_1 = \bigcap_i \bar{N}_i \cap \bigcap_i \bar{M}_i$. Then since N_1 is the intersection of finitely many normal subgroups of finite index in A_1, N_1 is normal and of finite index in A_1, and since $\bar{N}_i \supset N_1$, $\bar{M}_i \supset N_1$, $\varphi_{m_i}^{-1}(b_i)\, B_1 \cap N_1$ and $\varphi_{m_i}^{-1}(a_i)\ B_1 \cap N_1$ are empty. Let $N_i = \varphi_i(N_1)$ so that $A_i \supset N_i$. Since the φ are isomorphisms, N_i is normal in A_i, A_i/N_i is finite and $b_iB_{m_i} \cap N_{m_i}$ $\quad a_iB_{n_i} \cap N_{n_i}$ are empty.

Let $X(G)$ be the free product of A_i/N_i with $B_i/B_i \cap N_i$ amalmagated, and let X be the natural mapping. Then $X(g_i)$ has the same syllable length in $X(G)$ as it has in G. Then $X(g_1) \neq X(g_2)^t$ for all integers and $X(G)$ is Π_c.

Case II. Now consider g_2 of syllable length 1 and g_1 of syllable length 1, and suppose $g_1 \neq g_2{}^t$ for all t. If g_1 and g_2 are in the same factor A_i, there is a mapping $X : G \rightarrow A_1$ which is φ_i^{-1} on g_1 and g_2 so that $X(g_1) \neq X(g_2)^t$ for all t and A_1 is Π_c. If g_1 and g_2 are in different factors, neither can be in the amalgamated subgroup, so that the process described in the discussion of Case I will yield an image group $X(G)$ in which $X(g_1) \neq X(g_2)^t$ for all t, since $X(g_1)$ and $X(g_2)$ lie in different factors, so that $X(g_2)^t$ is either only in the factor of g_2 or is in the amalgamated subgroup of $X(G)$. But $X(g_1)$ is not in the amalgamated subgroup of $X(G)$.

Case III. Let g_1 and g_2 be elements of G_1 with g_2 cyclically reduced, of syllable length greater than one, $g_1 \neq g_2{}^t$ for all t and the syllable length of g_1 divisible by the syllable length of g_2. In particular, let $g_2 = a_1 \cdots a_m$; $g_1 = b_1 \cdots b_{km}$. Let N_1 be constructed as in Case I, which is possible since no a_i or b_i is in the amalgamated subgroup. Since $g_1 \neq g_2^{\pm k}$ at least one equation in each of the sets of equation of Lemma 4, b, is violated for both $+k$

and $-k$. Suppose $a_i^{-1}c_{i-1}b_i = c_i$ is the first of the equations of Lemma 2 for $+k$ to be violated in the sense that either c_i is not in the amalgamation or $c_{km} \neq 1$. Let $c_i \in A_p$, and let M_1 be a normal subgroup of finite index in A_1 such that $\varphi_p^{-1}(c_i)\, B_1 \cap M_1$ is empty if $i \neq km$ or $\varphi_p^{-1}(c_i) \notin M_1$ if $i = km$. Let M_2 be the similarly defined subgroup for $-k$. Let $U_1 = N_1 \cap M_1 \cap M_2$ and let $U_i = \varphi_i(U_1)$. Then U_i is normal and of finite index in A_i. Let $X(G)$ be the free product of A_i/U_i with $B_i/U_i \cap B_i$ amalgamated. Let X be the natural mapping. Then since $N_i \supset U_i$, $X(g_1)$ has length km, $X(g_2)$ has length m so that $X(g_1) = X(g_2)^t$ only if $t = \pm k$. But since

$$M_1 \supset U_1,\ M_2 \supset U_1,\ X(g_1) \neq X(g_2)^{\pm k}.$$

Thus $X(g_1) \neq X(g_2)^t$ for all t and $X(G)$ is Π_c.

Case IV. Let g_1 and g_2 be elements of G with $g_1 \neq g_2{}^t$ for all integers t. If g_2 is not cyclically reduced, there is an element g of G such $g^{-1}g_2g$ is cyclically reduced and $g^{-1}g_1g \neq (g^{-1}g_2g)^t$ for all t. Thus there is a Π_c group $X(G)$ such that $X(g^{-1}g_1g) \neq X(g^{-1}g_2g)^t$ for all t, and hence $X(g_1) \neq X(g_2)^t$ for all t. If g_2 is cyclically reduced, the existence of $X(G)$ follows from cases I, II, and III. Clearly, G must be Π_c. Since $X(G)$ is Π_c, there is in every case a mapping ξ from $X(G)$ onto a finite group such that $\xi X(g_1) \neq \xi X(g_2)^t$ for all t. If K is the kernel of ξX, $g_1 \not\equiv g_2{}^t \bmod K$ for all t and K is of finite index in G. It also follows that:

If A has the property ascribed to free groups in Lemma 1, the free product of groups isomorphic to A with a single finitely generated amalgamated subgroup is Π_c.

References

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